Quasiconformal Mappings and Solutions of the Dispersionless KP hierarchy *

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Abstract

A $\bar{\partial}$-formalism for studying dispersionless integrable hierarchies is applied to the dKP hierarchy. Connections with the theory of quasiconformal mappings on the plane are described and some clases of explicit solutions of the dKP hierarchy are presented.

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1 Introduction

Dispersionless or quasiclassical integrable hierarchies form an important part of the theory of integrable systems . They are the main ingredients of various approaches to different problems arising in physics and applied mathematics (see e.g. [1]-[10]) . In this sense, it is worth-mentioning their connections [11, 12] with some classical problems of the theory of conformal maps. Furthermore, we have recently proposed a $\bar{\partial}$-method [13]-[14] for studying dispersionless integrable hierarchies which reveals an intimate relation between these hierarchies and the theory of quasiconformal mappings [15]-[18].

The basic element of our analysis is the nonlinear $\bar{\partial}$-equation

$$S_{\bar{z}} = W\Big(z, \bar{z}, S_z\Big), \tag{1}$$

where $z \in \mathbb{C}$, $S(z, \bar{z}, \boldsymbol{t})$ is a complex-valued function depending on an infinite set $\boldsymbol{t}$ of parameters (times), $S_{\bar{z}} := \frac{\partial S}{\partial \bar{z}}$, $S_z := \frac{\partial S}{\partial z}$ and W is an appropriate function of $z, \bar{z}$ and S_z.

As a consequence of (1) the first-order derivatives of S with respect to the $\boldsymbol{t}$ parameters satisfy the family of Beltrami equations

$$f_{\bar{z}} = \mu(z, \boldsymbol{t}) f_z, \tag{2}$$

where

$$\mu := W'(z, \bar{z}, S_z), \quad W' = W_\xi(z, \bar{z}, \xi). \tag{3}$$

This fact provides us with the link between the $\bar{\partial}$-method and the theory of quasiconformal mappings, in which Beltrami equation is of fundamental importance. It should be mentioned that equation (1), in turn, is also well-known in the theory of quasiconformal mappings (see e.g. [19]-[20]).

Our method is based on the determination of solutions of (1) by means of the classical schemes for solving first-order PDE's of Hamilton-Jacobi type . The objective of the present paper is to illustrate our approach by presenting some exact explicit solutions of the dKP hierarchy. We also prove that the simplest of this solutions (Example 1 of Section 4) can not be reached by the standard methods based on the hodograph transformation technique [3]-[4].

2 The dKP hierarchy

The dKP hierarchy can be introduced as the following *classical version* of the Lax-pairs equations of the standard KP theory [1]-[9]

$$\frac{\partial z}{\partial t_n} = \{\Omega_n, z\}, \quad \Omega_n(p, \boldsymbol{t}) := (z^n)_+, \quad n \geq 1. \tag{4}$$

Here $z = z(p, \boldsymbol{t})$ is a complex function depending on a complex variable p and an infinite set $\boldsymbol{t} := (t_1, t_2, \dots)$ of complex parameters, which is assumed to posses a Laurent expansion of the form

$$z = p + \sum_{n \geq 1} \frac{a_n(\boldsymbol{t})}{p^n}, \tag{5}$$

near $p \to \infty$. We denote by $(z^n)_+$ the polynomial part of the expansion of z^n in powers of p

$$(z)_+ = p, \quad (z^2)_+ = p^2 + 2a_1, \quad (z^3)_+ = p^3 + 3p\, a_1 + 3a_2,$$

and the Poisson bracket is defined as

$$\{F, G\} := \frac{\partial F}{\partial p}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial p}, \quad x := t_1.$$

The compatibility conditions for (4) are of the form

$$\frac{\partial \Omega_m}{\partial t_n} - \frac{\partial \Omega_n}{\partial t_m} + \{\Omega_m, \Omega_n\} = 0, \quad m \neq n, \tag{6}$$

Two interesting examples of nonlinear equations of the dKP hierarchy are:

1) For $n = 2$, equation (4) leads to the Benney moment equations

$$\frac{\partial a_{n+1}}{\partial t} + \frac{\partial a_{n+2}}{\partial x} + n a_n \frac{\partial a_1}{\partial x} = 0, \quad t := -2t_2. \tag{7}$$

2) The compatibility equations (6) for $n = 2$ and $m = 3$ imply the dKP equation (Zabolotskaya–Khokhlov equation)

$$(u_t - \frac{3}{2} u u_x)_x = \frac{3}{4} u_{yy}, \quad u := 2a_1,\ t := t_3,\ y := t_2. \tag{8}$$

From (6) it follows [6] that for any solution $z = z(p, \boldsymbol{t})$ of the dKP hierarchy, there exists an associated function $S = S(z, \boldsymbol{t})$, such that

$$\frac{\partial S(z, \boldsymbol{t})}{\partial t_n} = \Omega_n(p(z, \boldsymbol{t}), \boldsymbol{t}), \quad n \geq 1. \tag{9}$$

Here $p = p(z, \boldsymbol{t})$ is obtained by inverting the solution $z = z(p, \boldsymbol{t})$. Without loss of generality it can be assumed that S has a Laurent expansion

$$S(z, \boldsymbol{t}) = \sum_{n \geq 1} z^n t_n + \sum_{n \geq 1} \frac{S_n(\boldsymbol{t})}{z^n}, \quad z \in \Gamma, \tag{10}$$

on a certain circle $\Gamma = \{z : |z| = r\}$. Observe that by setting $n = 1$ in (9) and using (10) one characterizes $p = p(z, \boldsymbol{t})$ in the form

$$p = \frac{\partial S(z, \boldsymbol{t})}{\partial x} = z + \sum_{n \geq 1} \frac{b_n(\boldsymbol{t})}{z^n}, \quad b_n := \frac{\partial S_n}{\partial x}, \tag{11}$$

Reciprocally, given a function $S = S(z, \boldsymbol{t})$ which satisfies (9) and (10), it can be proved [6] that the inverse function $z = z(p, \boldsymbol{t})$ of the function $p = p(z, \boldsymbol{t})$ of (11) determines a solution of the dKP hierarchy.

Henceforth, functions verifying conditions (9) and (10) will be referred to as *S-functions* of the dKP hierarchy. They turn out to be related to the τ-functions [6] according to

$$S(z, \boldsymbol{t}) = \sum_{n \geq 1} z^n t_n + \sum_{n \geq 1} \frac{1}{n\, z^n} \frac{\partial \ln \tau(\boldsymbol{t})}{\partial t_n}, \quad z \in \Gamma.$$

We notice that the system (9) for a dKP S-function constitutes a set of compatible Hamilton-Jacobi type equations

$$\frac{\partial S}{\partial t_n} = \Omega_n\Big(\frac{\partial S}{\partial x},\ \boldsymbol{t}\Big), \quad n \geq 2, \tag{12}$$

which represents the *semiclassical limit* of the linear system for the wave function of the standard KP hierarchy.

Several methods for constructing solutions of the dKP hierarchy through S-functions have been devised [3, 4, 6, 8]. We will be here concerned with the $\bar{\partial}$-method proposed in [13]-[14], which aims to characterize S-functions by means of solutions of $\bar{\partial}$-equations of the form (1). A main feature of

this approach is that the symmetries of (1) (first order variations $f := \delta S$) are determined by the family of Beltrami equations (2)-(3). This property implies, in particular, that all the first-order derivatives $\frac{\partial S}{\partial t_n}$ of a solution of (1) satisfy (2).

Given any fixed local solution f of a Beltrami equation, if f has non-zero Jacobian at a certain point z_0, then all the smooth local solutions F near z_0 are given by analytic functions $F = F(f)$ of f . This property suggests that, under *appropriate conditions*, solutions S of a $\bar{\partial}$-equation (1) which admit expansions of the form (10), satisfy conditions (12) as well. Therefore, they provide S-functions for the dKP hierarchy. In what follows a rigorous basis for this scheme is proposed.

3 Quasiconformal mappings

Quasiconformal mappings are a natural and very rich extension of the concept of conformal mappings. For the sake of convenience we remind here some of their basic properties (see e.g. [13]-[18]).

Let $\mu = \mu(z)$ be a measurable function on a domain G of the complex plane such that for some $0 < k < 1$ it verifies $|\mu(z)| < k$ almost everywhere in G . Then, a function $f = f(z)$ is said to be a *quasiconformal mapping (qc-mapping) with complex dilatation* μ in G if

i) f is a homeomorphism $f: D \to D'$

ii) f is a generalized solution of the linear Beltrami equation

$$f_{\bar{z}} = \mu f_z, \tag{13}$$

on D, with locally square-integrable partial derivatives $f_{\bar{z}}$ and f_z.

The properties of solutions of the Beltrami equation (13) are rather well studied (see e.g. [16]). Some of them are particularly important for our discussion. Before presenting these results we introduce the Calder'on-Zygmund operator [16]

$$(Th)(z) := \frac{1}{2\pi\imath} \iint_{\mathbb{C}} \frac{h(z')}{(z'-z)^2} \,\mathrm{d}\, z' \wedge \mathrm{d}\, \bar{z}', \tag{14}$$

where the integral is taken in the sense of the Cauchy principal value. Then one has the following fundamental result [16]:

Theorem 1. *For any $p \geq 2$ the operator T defines a bounded operator in $L^p(\mathbb{C})$. Moreover, $||T||_p$ is continuos with respect to p and satisfies*

$$\lim_{p\to 2} ||T||_p = 1.$$

As a consequence of this theorem it follows that for any $0 \leq k < 1$ there exists $\delta(k) > 0$ such that

$$k||T||_p < 1,$$

for all $2 < p < 2 + \delta(k)$.

The next theorem provides us with a property of qc-mappings which will be useful in the $\bar{\partial}$-method .

Theorem 2. *Let μ be a measurable function with compact support inside the circle $|z| < R$ and such that $||\mu||_\infty < k < 1$. Then, for any $p > 2$ such that $k||T||_p < 1$, it follows that the only generalized solution of Beltrami equation* (5) *verifying*

$$f(z) = O(\frac{1}{z}), \quad z \to \infty, \tag{15}$$

and $f_{\bar{z}}, f_z \in L^p(\mathbb{C})$ is $f \equiv 0$.

This result is the uniqueness part of the existence theorem for the so-called normal solutions of Beltrami equations [15]-[17]. Its proof relies on the fact that the operator T represents the action of $\partial_z \partial_{\bar{z}}^{-1}$ on $L^p(\mathbb{C})$, so that under the hypothesis of the theorem we have that Beltrami equation for f becomes the integral equation

$$\phi - \mu T\phi = 0, \quad \phi := f_{\bar{z}},$$

on $L^p(\mathbb{C})$. Thus, by taking into account that $||\mu T||_p \leq k||T||_p < 1$, we have $\phi \equiv 0$ and then, in view of (15), $f \equiv 0$.

Let us go back to our $\bar{\partial}$-equation (1) and let us assume that $W(z, \bar{z}, S_z)$ vanishes for all z outside a certain circle $\Gamma = \{z : |z| = r\}$. Suppose that we are able to find a solution $S = S(z, \bar{z}, \boldsymbol{t})$ of (1) in the disk $D = \{z : |z| < r\}$ with a boundary value $S|_\Gamma := S(z, \frac{r^2}{z}, \boldsymbol{t})$ of the form (10), and such that for a certain $0 < k < 1$ the set

$$\Omega := \{\boldsymbol{t} : \sup_{z\in D} |W'(z, \bar{z}, S_z(z, \boldsymbol{t})| \leq k\},$$

is not empty. In this case we can apply Theorem 2 to the Beltrami equation (2)-(3). Moreover, by taking into account that

$$\frac{\partial S}{\partial t_n} = z^n + \sum_{m\geq 1} \frac{\partial S_m(\boldsymbol{t})}{\partial t_n} z^{-m}, \quad z \in \Gamma,$$

these functions can be continuously extended from D to analytic functions outside Γ. Thus, they become solutions of (2)-(3) on the whole complex plane. On the other hand, it is clear that

$$\frac{\partial S}{\partial t_n} - (z^n)_+ = \frac{\partial S}{\partial t_n} - \Omega_n\Big(\frac{\partial S}{\partial x},\ \boldsymbol{t}\Big) = O(\frac{1}{z}), \quad z \to \infty,$$

so that from Theorem 2 we conclude that (9) is satisfied and, consequently, $S = S(z, \bar{z}, \boldsymbol{t})$ determines an S-function of the dKP hierarchy for $\boldsymbol{t} \in \Omega$.

4 Solutions of the dKP hierarchy

In order to construct explicit solutions of the dKP hierarchy we consider $\bar{\partial}$-equations of the form

$$S_{\bar{z}} = \theta(r - |z|)V\Big(z, \bar{z}, S_z\Big), \tag{16}$$

where $r > 0$, $\theta(\xi)$ is the usual Heaviside function and V is an analytic function of z, $\bar{z}$ and S_z. Our scheme of solution is as follows

1) Firstly, we generate solutions $S = S(z, \boldsymbol{a})$ of (16)

$$S_{\bar{z}} = V\Big(z, \bar{z}, S_z\Big), \quad |z| < r, \tag{17}$$

depending on a set of free parameters $\boldsymbol{a} := (a_0, a_1, \dots)$.

2) Then we select those solutions whose boundary value on $\Gamma = \{z : |z| = r\}$ is of the form (10).

Equation (17) is a PDE of Hamilton-Jacobi type , so that several powerful methods for generating solutions are available. For example, if $V = V(S_z)$ depends only on S_z then (17) implies

$$m_{\bar{z}} = V_m(m)m_z, \quad m := S_z.$$

This equation can be solved at once by applying the methods of characteristics. So the general solution (17) is implicitly characterized by

$$
\begin{aligned}
S &= V(m)\bar{z} + m\,z - f(m),\\
V_m\bar{z} + z &= f_m(m),
\end{aligned}
\tag{18}
$$

where $f = f(m)$ is an arbitrary function. Notice that according to the second equation in (18), we have

$$f_m(m_0) = z, \quad m_0 := m(z,\bar{z})|_{\bar{z}=0},$$

so that $f_m(m_0)$ is the inverse function of $m_0 = m_0(z)$.

As we are aiming to get explicit solutions some simplifying assumptions are required. For instance, we consider cases in which only a finite set of $N+1$ parameters $\boldsymbol{a} = (a_0, a_1, \dots, a_N)$ are involved. Therefore we are facing the problem of selecting solutions S in which no terms $z^n t_n$ with $n > N$ in (10) appear. Other type of solutions of the $\bar{\partial}$-equation would have time-parameters t_n $(n > N)$ which are functions of $(t_1, \dots, t_N)$ and no solution of the dKP hierarchy would arise in that way.

In order to explore the possible favorable cases, let us consider the class of $\bar{\partial}$-equations (17) of the form

$$S_{\bar{z}} = \bar{z}^s \sum_{m\geq 0}^{M} p_m(z)(S_z)^m, \quad |z| < r, \tag{19}$$

where $s \geq 0$, $M \geq 2$, the coefficients $p_m = p_m(z)$ are polynomials in z and $p_M \not\equiv 0$. Let us look for a series solution of (19) of the form

$$S = \sum_{n\geq 0} c_n(z)\bar{z}^{n(s+1)}, \tag{20}$$

with c_0 being set as an arbitrary N-degree polynomial $(N \geq 2)$

$$c_0(z) = \sum_{n=0}^{N} a_n z^n, \quad a_N \neq 0. \tag{21}$$

Substitution of (20) in (19) provides the recursion relation

$$c_{n+1} = \frac{1}{(n+1)(s+1)} \sum_{m\geq 0}^{M} p_m(z)\Big(\sum_{r_1+\cdots r_m=n} c'_{r_1}\cdots c'_{r_m}\Big), \quad n \geq 0, \tag{22}$$

which shows that all coefficients $c_n(z)$ of (20) are polynomials. We need to know their degree in order to be able to examine the form of S on the boundary $|z| = r$.

Lemma 1. *If the degrees of the coefficients p_m in (19) verify*

$$deg(p_m) \leq (M-m)(N-1), \quad m = 0, 1, \dots, M,$$

then

$$deg(c_n) = n[M(N-1) - N] + N, \quad n \geq 0. \tag{23}$$

Proof

We apply the induction principle. It is obvious that (23) is true for $n = 0$. Suppose now that it holds for $n' \leq n$, and consider the terms in the expression (22) for c_{n+1}. By taking into account that $r_1 + \cdots + r_m = n$ we get

$$\begin{aligned} deg(p_m\, c'_{r_1} \cdots c'_{r_m}) &= \sum_{i=1}^{m} \Big[r_i \Big(M(N-1) - N \Big) + N - 1 \Big] + deg(p_m) \\ &= n\Big[M(N-1) - N \Big] + m(N-1) + deg(p_m) \\ &\leq n\Big[M(N-1) - N \Big] + M(N-1) \\ &= (n+1)\Big[M(N-1) - N \big] + N. \end{aligned}$$

Moreover, as p_M is a non-zero constant, it is clear that the corresponding terms in (22) verify

$$\begin{aligned} deg(p_M\, c'_{r_1} \cdots c'_{r_M}) &= n\Big[M(N-1) - N \Big] + M(N-1) \\ &= (n+1)\Big[M(N-1) - N \Big] + N. \end{aligned}$$

Thus, one concludes that (23) is verified for c_{n+1} too and, therefore, the statement is proved (QED).

If we assume that the series (20) converges for a certain $r \geq 0$, then under the hypothesis of the above lemma the continuous extension of S to the boundary $|z| = r$ is of the form

$$S = \sum_{n\geq 0} r^{2n(s+1)} \frac{c_n(z)}{z^{n(s+1)}}, \quad \frac{c_n(z)}{z^{n(s+1)}} = O\Big(z^{d_n} \Big), \tag{24}$$

where

$$d_n = n[M(N-1) - N - s - 1] + N. \qquad (25)$$

Since the solution corresponding to (21) depends on $N+1$ free parameters $(a_0, \dots, a_N)$, only those cases for which $d_n \leq N$ for all $n \geq 0$ are of interest. From (25) it is obvious that this happens only if

$$N \leq \frac{M+s+1}{M-1}. \qquad (26)$$

For example if we set $s = 0$, this means that we have only the three possibilities exhibited in the following table:

(M, N)	$V(z, S_z)$	$S\vert_{\bar{z}=0}$
(2,2)	$\alpha(S_z)^2 + (\sum_{i=0}^{1} \beta_i z^i) S_z + \sum_{i=0}^{2} \gamma_i z^i$	$\sum_{i=0}^{2} a_i z^i$
(3,2)	$\alpha(S_z)^3 + (\sum_{i=0}^{1} \beta_i z^i)(S_z)^2 + (\sum_{i=0}^{2} \gamma_i z^i) S_z + \sum_{i=0}^{3} \eta_i z^i$	$\sum_{i=0}^{2} a_i z^i$
(2,3)	$\alpha(S_z)^2 + (\sum_{i=0}^{2} \beta_i z^i) S_z + \sum_{i=0}^{4} \gamma_i z^i$	$\sum_{i=0}^{3} a_i z^i$

Example 1. The simplest case in the class $(2, 2)$ corresponds to

$$S_{\bar{z}} = \theta(1 - |z|)(S_z)^2, \qquad (27)$$

with $S|_{\bar{z}=0}$ being a quadratic polynomial. This yields to

$$S = \begin{cases} \frac{1}{2}\frac{(z-b)^2}{a-2\bar{z}} - c, & |z| \leq 1 \\ \\ \frac{1}{2}\frac{z(z-b)^2}{az-2} - c, & |z| \geq 1. \end{cases}$$

Notice that the regularity of S inside the unit circle requires

$$|a| > 2. \qquad (28)$$

On the boundary $|z| = 1$ we have

$$S = \frac{1}{2a}z^2 + (\frac{1}{a^2} - \frac{b}{a})z + \frac{2}{a^3} + \frac{b^2}{2a} - \frac{2b}{a^2} - c + O(\frac{1}{z}).$$

So, in order to fit with the required form of an S-function of the dKP hierarchy we have to identify

$$x = \frac{1}{a^2} - \frac{b}{a}, \quad t_2 = \frac{1}{2a}, \quad c = \frac{2}{a^3} + \frac{b^2}{2a} - \frac{2b}{a^2}.$$

On the other hand, the complex dilatation for the corresponding Beltrami equation (1)-(2) is given by

$$\mu(z,\bar{z}) := 2\theta(1-|z|)\frac{z-b}{a-2\bar{z}}. \tag{29}$$

We observe that the following bound follows

$$|\mu(z,\bar{z})| < 2\frac{|b|+1}{|a|-2}, \quad z \in \mathbb{C}.$$

In this way, for any $0<k<1$ we have $|\mu(z)| \leq k$ provided $k|a| > 2(|b|+k+1)$. Thus, there is a non empty domain in the space of parameters on which the Beltrami equation (1)-(2) satisfies the conditions assumed in our discussion of Section 3.

It follows that

$$p := S_x = \frac{z^2 - 4t_2 z + 2(x+4t_2^2)}{z-4t_2}, \quad |z| = 1,$$

so one gets the following solution of the t_2-flow (Benney flow) of the dKP hierarchy

$$z = \frac{p}{2} + 2t_2 + \sqrt{(\frac{p}{2} - 2t_2)^2 - 2x - 8t_2^2}. \tag{30}$$

We notice that this solution depends on the time-parameters trough the pair of functions $u_1 := 2t_2,\ u_2 := -2x - 8t_2^2$. Indeed, we can rewrite it as

$$z = \frac{p}{2} + u_1 + \sqrt{(\frac{p}{2} - u_1)^2 + u_2}.$$

In case this solution could be obtained by the hodograph methods [3]-[4], it would correspond to a reduction of the dKP hierarchy in which the functions $\boldsymbol{u} = (u_1, u_2)$ would satisfy a diagonalizable hydrodynamic type system with Riemann invariants provided by the zeros of the function $\frac{\partial z(p,\boldsymbol{u})}{\partial p}$. But this function has no zeros for $u_2 \neq 0$ as

$$\frac{\partial z(p,\boldsymbol{u})}{\partial p} = \frac{\frac{p}{2} - u_1 + \sqrt{(\frac{p}{2}-u_1)^2 + u_2}}{2\sqrt{(\frac{p}{2}-u_1)^2 + u_2}}.$$

Therefore, we conclude that the solution (30) of the dKP hierarchy can not be reached from the hodograph technique approach.

Example 2. The $\bar{\partial}$-equation

$$S_{\bar{z}} = \theta(1-|z|)(S_z)^3, \tag{31}$$

with $S|_{\bar{z}=0}$ being a quadratic polynomial, is an example of the $(3,2)$ case. One finds the S-function

$$S = \bar{z}m^3 + (z-b)m - \frac{a}{2}m^2 - c, \quad |z|<1,$$

where

$$m = \frac{1}{6\bar{z}}\Big(a - \sqrt{a^2 - 12(z-b)\bar{z}}\Big).$$

The defining relations for the dKP parameters (x, t_2) are

$$x = \frac{b}{6}\Big(\sqrt{a^2-12} - a\Big), \; t_2 = \frac{1}{108}\Big(18a - a^3 + (a^2-12)^{3/2}\Big). \tag{32}$$

The corresponding solution of the t_2-flow of the dKP hierarchy can be written as

$$z = a(a-\sqrt{a^2-12})\frac{p}{12} + \frac{3x}{\sqrt{a^2-12}} + \frac{1}{12}\Big[\Big(a(a-\sqrt{a^2-12})p + \frac{36x}{\sqrt{a^2-12}}\Big)^2$$

$$- 12\Big((a-\sqrt{a^2-12})p + \frac{3(a+\sqrt{a^2-12})x}{\sqrt{a^2-12}}\Big)^2\Big]^{\frac{1}{2}},$$

where $a = a(t_2)$ is to be obtained from (32).

Example 3. The class $(2,3)$ is the most interesting one as it provides solutions of the dKP hierarchy depending of (x, t_2, t_3). Let us consider

$$S_{\bar{z}} = \theta(1-|z|)(S_z)^2, \tag{33}$$

with a cubic polynomial $S|_{\bar{z}=0}$. If we take $m_0 := S_z|_{\bar{z}} = az^2 + bz + c$, then from (18) we get

$$\begin{aligned} f(m) &= -\frac{b}{2a}m + \frac{1}{12a^2}(4am + b^2 - 4ac)^{\frac{3}{2}} + d \\ &= -\frac{b}{2a}m + \frac{1}{12a^2}(4a\bar{z}m + 2az + b)^3 + d, \end{aligned} \tag{34}$$

and

$$m = \frac{1}{8\bar{z}^2}\Big(\frac{1}{a} - 4(z+\frac{b}{2a})\bar{z} - \sqrt{\frac{4}{a}(\frac{b^2}{a} - 4c)\bar{z}^2 - \frac{8}{a}(z+\frac{b}{2a})\bar{z} + \frac{1}{a^2}}\ \Big). \qquad (35)$$

Hence we have

$$S = (z+\frac{b}{2a})m + \bar{z}m^2 - \frac{1}{12a^2}(4a\bar{z}m + 2az + b)^3 - d, \quad |z| < 1. \qquad (36)$$

It is clear that in order to ensure S to be continuous we have to require [14]

$$\frac{4}{a}(\frac{b^2}{a} - 4c)\bar{z}^2 - \frac{8}{a}(z+\frac{b}{2a})\bar{z} + \frac{1}{a^2} \neq 0, \quad |z| < 1.$$

Notice also that S is regular at the origin as

$$\lim_{z\to 0} m = c.$$

Let us outline the calculation of $u = -2\frac{\partial S_1}{\partial x}$. We need to compute the first few terms of the expansion of S on $|z| = 1$. To do that it is helpful to use the following identity

$$S_z = m - \Big(\frac{m}{z}\Big)^2, \quad |z| = 1. \qquad (37)$$

Thus, from the expansion of m on $|z| = 1$ and by setting

$$S_z = 3t_3z^2 + 2t_2z + x - \frac{S_1}{z^2} + \dots, \quad |z| = 1,$$

in (37), we get

$$3t_3 = -\frac{3}{4} + \frac{3}{8a} + \frac{1}{32a^2}\Big((1-8a)^{3/2} - 1\Big), \qquad 2t_2 = \frac{b}{8a^2}(1 - 4a - \sqrt{1-8a}),$$

$$x = -\frac{b^2}{8a^2}\Big(1 + \frac{4a-1}{2\sqrt{1-8a}}\Big) + \frac{c}{4a}(1 - \sqrt{1-8a}),$$

and

$$S_1 = \frac{\Big(2b^2 + (1-8a)c\Big)^2}{(1-8a)^{5/2}}. \qquad (38)$$

From these expressions it can be computed that $u = -2\frac{\partial S_1}{\partial x}$ is given by $(t := t_3, y := t_2)$

$$u = \frac{4\left(5 - 12t + 4\sqrt{1-12t}\right)^2\left((-1+12t)x - 4y^2\right)}{3(1+4t)(1-12t+2\sqrt{1-12t})^3}, \tag{39}$$

which satisfies (8). We notice that according to [10], (39) belongs to the class of solutions of the dKP equation which yield Einstein-Weyl structures conformal to Einstein metrics .

More general cases corresponding to (19) with $s \neq 0$ and z-dependence on its right-hand side will be considered elsewhere [21].

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